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Dimensional Effects on Solitonic Matter and Optical Waves with Normal and Anomalous Dispersion

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Abstract. We investigate bright and dark solitons with anomalous or normal dispersion and under transverse harmonic confinement. In matter waves, positive atomic mass implies anomalous dispersion (kinetic spreading) while negative mass gives normal dispersion (kinetic shrinking). We find that, contrary to the strictly one-dimensional case, the axial and transverse profiles of these solitons crucially depend on the strength of the nonlinearity and on their dispersive properties. In particular, we show that, like bright solitons with anomalous dispersion, also dark solitons with normal dispersion disappear at a critical axial density. Our predictions are useful for the study of atomic matter waves in Bose-Einstein condensates and also for optical bullets in inhomogeneous Kerr media.

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1. Introduction

Matter waves made of Bose condensed alkali-metal atoms [1] and optical waves in nonlinear Kerr media [2] are accurately described by a three-dimensional nonlinear Schrödinger equation (3D NLSE). In fact, the NLSE is a unifying theoretical tool of nonlinear optics and of the new field of research called nonlinear atom optics [3, 4]. Many experiments have been devoted to the study of solitary waves in both matter waves [5, 6, 7, 8, 9] and optical waves [10]. Bright and dark solitons are usually analyzed in quasi-1D configurations, where analytical solutions of the 1D cubic NLSE are available [11, 12]. On the other hand, it has been shown that dimensional effects can be quite important: under transverse confinement atomic bright solitons with anomalous dispersion, i.e. kinetic spreading, become unstable if the transverse confinement is not sufficiently strong [13].

In this paper we analyze bright and dark solitons with anomalous and normal dispersion (kinetic shrinking) taking into account the transverse width of the solitary wave. We introduce a non-polynomial Schrödinger equation, which extends that we

derived some years ago [14] by including the case of the normal dispersion. In this way, we find new analytical solutions for 3D bright solitons under transverse harmonic confinement and show that, contrary to the strictly 1D case, 3D black and gray solitons with normal dispersion exist only below a critical axial density. Our predictions can be tested not only with matter waves in a periodic optical potential along the axial direction [15] but also with optical bullets in a inhomogeneous graded-index Kerr medium [16].

2. Matter waves and optical pulses

The 3D cubic nonlinear Schrödinger equation (3D CNLSE), which describes both matter waves in a Bose-Einstein condensate and optical pulses in a inhomogeneous nonlinear Kerr medium, is given by

$$i\frac{\partial\psi}{\partial t} = \Big[-\frac{1}{2}\nabla_{\perp}^2 - \frac{\delta_D}{2}\frac{\partial^2}{\partial z^2} + \frac{1}{2}(x^2+y^2) - \delta_N 2\pi g|\psi|^2 \Big]\psi \ , \tag{1}$$

where $g > 0$ and the parameters $\delta_D = \pm 1$ and $\delta_N = \pm 1$, according as whether the axial kinetic term has anomalous ($\delta_D = 1$) or normal ($\delta_D = -1$) dispersion, and the nonlinear cubic term is self-focusing ($\delta_N = 1$) or self-defocusing ($\delta_N = -1$). In the context of nonlinear atom optics, the field $\psi(\mathbf{r}, t)$ of Eq. (1) is the macroscopic wave function of the Bose-Einstein condensate. In Eq. (1) the time t is in units of $1/\omega_{\perp}$, where $\omega_{\perp}$ is the frequency of the transverse harmonic potential. The lengths are in units of the characteristic harmonic length $a_{\perp} = \sqrt{\hbar/(m\omega_{\perp})}$, where m is the atomic mass. The mass m is the true atomic mass in the absence of an axial confinement ($\delta_D = 1$) [13, 14], but it is instead the modulus of a negative effective mass which takes into account the effect of periodic axial potential when atoms are around the upper band edge ($\delta_D = -1$) [15, 16, 17]. The strength g of the cubic nonlinearity is proportional to the modulus of the s-wave scattering length of the inter-atomic potential and its sign is given by the parameter δ_N: $\delta_N = 1$ corresponds to an effective interatomic attraction and $\delta_N = -1$ corresponds to an effective repulsion[13, 14, 15, 16].

In the context of nonlinear guided wave optics, the field $\psi(\mathbf{r}, t)$ of Eq. (1) represents instead the envelope of the electric field oscillating at a fixed frequency. Both the cubic nonlinearity and the transverse harmonic confinement model the refrective index of an inhomogeneous Kerr medium [17]. Note that in describing an optical pulse in fibers with Eq. (1) the two variables t and z have an inverted meaning: the variable t is the axial coordinate of the electric field while the variable z is the time coordinate of the electric field. The scaling of these variables is discussed, for instance, in Ref. [17], where g is set equal to $1/(2\pi)$ and the field $\psi(\mathbf{r}, t)$ is normalized such that $\int |\psi(\mathbf{r}, t)|^2 d^3\mathbf{r}$ represents the constant pulse energy.

To study the 3D CNLSE time-dependent variational methods are often used [2, 10, 12, 18]. The 3D CNLSE is the Euler-Lagrange equation obtained by minimizing the following action functional

$$A = \int \psi^* \Big[i\frac{\partial}{\partial t} + \frac{1}{2}\nabla_{\perp}^2 + \frac{\delta_D}{2}\frac{\partial^2}{\partial z^2} - \frac{1}{2}(x^2+y^2) + \delta_N\, \pi g|\psi|^2 \Big]\psi \ d^3\mathbf{r}\ dt \ . \tag{2}$$

Some years ago we proposed [14] a variational ansatz of the field $\psi(\mathbf{r},t)$ for the Eqs. (1) and (2) with $\delta_D = 1$. The resulting effective non-polynomial Schrödinger equation is very accurate in reproducing the numerical results of the 3D CNLSE with $\delta_D = 1$ [14, 19]. This effective equation has been used also to model the data of various experiments [20]. Here we use the same approach by considering also the case with $\delta_D = -1$. The variational ansatz is

$$\psi(\mathbf{r},t) = f(z,t)\frac{1}{\pi^{1/2}\sigma(z,t)^2} exp\left(\frac{x^2+y^2}{2\sigma(z,t)^2}\right) , \tag{3}$$

where $f(z,t)$ is the axial wave function and $\sigma(z,t)$ is the width of the transverse Gaussian wave function. By inserting Eq. (3) into Eq. (2) and integrating along x and y, the resulting effective action functional depends on two fields: $f(z,t)$ and $\sigma(z,t)$. Neglecting the derivatives of $\sigma(z,t)$ we find that the Euler-Lagrange equation of the axial wavefunction $f(z,t)$ given by

$$i\frac{\partial f}{\partial t} = \Big[-\frac{\delta_D}{2}\frac{\partial^2}{\partial z^2} - \delta_N \frac{g}{\sigma^2}|f|^2 + \frac{1}{2}\Big(\frac{1}{\sigma^2}+\sigma^2\Big)\Big] f , \tag{4}$$

and the Euler-Lagrange equation of the transverse width $\sigma(z,t)$ is instead given by

$$\sigma^2 = \sqrt{1-\delta_N\, g|f|^2} . \tag{5}$$

Inserting this formula into Eq. (4), we obtain a nonpolynomial Schrodinger equation (NPSE). In the weak-coupling limit $g|f|^2 \ll 1$, where $\sigma \simeq 1$, the NPSE reduces to the familiar 1D cubic nonlinear Schrödinger equation (1D CNLSE)

$$i\frac{\partial f}{\partial t} = \left[-\frac{\delta_D}{2}\frac{\partial^2}{\partial z^2} - \delta_N\, g|f|^2 \right] f , \tag{6}$$

where the additive constant 1 has been omitted because it does not affect the dynamics.

It is well known that the 1D CNLSE admits solitonic solutions. In particular, for $\delta_D \cdot \delta_N = 1$, i.e. for anomalous dispersion ($\delta_D = 1$) and self-focusing nonlinearity ($\delta_N = 1$) but also for normal dispersion ($\delta_D = -1$) and self-defocusing nonlinearity ($\delta_N = -1$), one finds the localized bright soliton solution

$$f(z,t) = \sqrt{\frac{g}{4}}\; Sech\left[\frac{g(z-vt)}{2}\right] e^{i\delta_D v(z-vt)} e^{i(v^2/(2\delta_D)-\mu)t} , \tag{7}$$

where $Sech[x]$ is the hyperbolic secant, v is the arbitrary velocity of propagation of the shape-invariant bright soliton and the parameter $\mu = -\delta_D g^2/8$ is obtained by the normalization condition of the field $f(z,t)$ to one [11, 12]. Instead, for $\delta_D \cdot \delta_N = -1$, i.e. for normal dispersion ($\delta_D = -1$) and self-focusing nonlinearity ($\delta_N = 1$) but also for anomalous dispersion ($\delta_D = 1$) and self-defocusing nonlinearity ($\delta_N = -1$), one finds the localized but not normalized dark soliton solution

$$f(z,t) = \sqrt{\frac{g\phi_\infty^2 - v^2}{g}}\, \Big(Tanh\left[(z-vt)\sqrt{g\phi_\infty^2 - v^2}\right] + i\frac{v\,\delta_D}{\sqrt{g\phi_\infty^2 - v^2}} \Big) e^{-i\mu t} , \tag{8}$$

where $Tanh[x]$ is the hyperbolic tangent, $\phi_\infty > 0$ is the constant value of the field amplitude $|f(z,t)|$ at infinity ($z \to \pm\infty$), and v is the velocity of propagation of the

dark soliton. The parameter $\mu = \delta_D g \phi_\infty^2$ is obtained by the asymptotic behavior of the field $f(z,t)$. For $0 < |v| < c_s$, where $c_s = \sqrt{g\phi_\infty^2}$ is the sound velocity, the minimum value of the field $f(z,t)$ is greater than zero and the wave is called gray soliton. If $v = 0$ (stationary dark soliton) the minimum of the field $f(z,t)$ is zero and the wave is called black soliton [11, 12]. The velocity c_s is the maximal velocity of the dark soliton, namely the velocity of a wave of infinitesimal amplitude, i.e. a sound wave, propagating in a medium of density $\rho_\infty = \phi_\infty^2$.

3. 3D bright and dark solitons

We have seen that for the 1D CNLSE the axial density profiles of bright and dark solitons do not depend on the sign of δ_D and δ_N, but only by their product $\delta_D \cdot \delta_N$ which discriminates between bright and dark solitons. It is important to stess that this property is valid only in the 1D limit. In fact, as previously shown, the 1D CNLSE is only the weak-coupling limit of 1D NPSE, that is derived from the 3D CNLSE. In the remaining part of the paper we show that, in general, for a fixed value of the strength g, 3D bright solitons ($\delta_D \cdot \delta_N = 1$) with anomalous dispersion ($\delta_D = 1$) and self-focusing nonlinearty ($\delta_N = 1$) do not have the same axial density profile of bright solitons with normal dispersion ($\delta_D = -1$) and self-defocusing nonlinearity ($\delta_N = -1$). The same happens for 3D dark solitons. In the remaining part of the paper we call 3D solitons the solitary solutions of the NPSE, which are good approximations of the exact solutions of the 3D CNLSE under transverse confinement.

To analyze 3D bright and dark solitons we start from the NPSE and set

$$f(z,t) = \phi(z - vt) e^{i\theta(z-vt)} e^{i(v^2/(2\delta_D)-\mu)t} , \tag{9}$$

where $\zeta = z - vt$ is the comoving coordinate of the soliton and both $\phi(\zeta)$ and $\theta(\zeta)$ are real fields. In this way we get an equation for $\phi(\zeta)$, namely

$$\left(\mu - \frac{v^2}{2\delta_D} + v\theta'\right)\phi = -\frac{\delta_D}{2}\left(\phi'' - \theta'^2\phi\right) + \frac{\phi - \delta_N (3/2) g\phi^3}{\sqrt{1-\delta_N\, g\phi^2}} , \tag{10}$$

and also an equation for the phase $\theta(\zeta)$, that is

$$v\,\phi' = \frac{\delta_D}{2}\left(\phi\,\theta'' + 2\phi'\theta'\right) . \tag{11}$$

Note that the prime means the derivative with respect to ζ; moreover, if $\phi = 0$ then from Eq. (11) one finds $\theta' = v/\delta_D$.

The two equations (10) and (11) are coupled but the second one can be written, for $\phi \neq 0$, as

$$v\frac{(\phi^2)'}{\phi} = \delta_D \frac{(\phi^2\theta')'}{\phi} , \tag{12}$$

from which we find

$$v\,\phi^2 = \delta_D\,\phi^2\theta' + \xi , \tag{13}$$

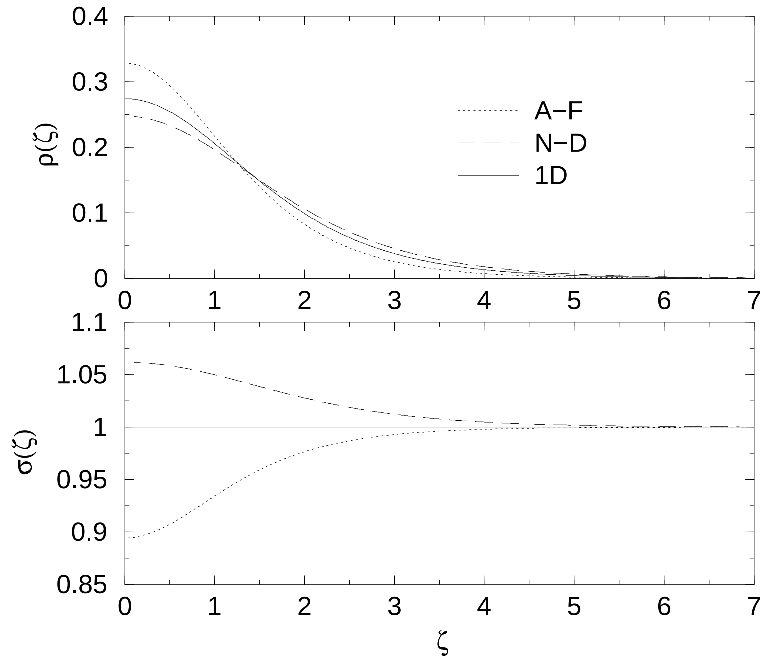

FIG. 1. Axial density profile $\rho(\zeta) = \phi(\zeta)^2$ and transverse width $\sigma(\zeta)$ of the bright soliton ($\delta_D \cdot \delta_N = 1$). Strength of the nonlinearity: $g = 1$. A-F means 3D bright soliton with anomalous dispersion ($\delta_D = 1$) and self-focusing nonlinearity ($\delta_N = 1$); N-D means 3D bright soliton with normal dispersion ($\delta_D = -1$) and self-defocusing nonlinearity ($\delta_N = -1$).

where the integration constant $\xi = 0$ for bright solitons and $\xi = v\phi_\infty^2$ for dark solitons, with ϕ_∞ the value of the field $\phi(\zeta)$ at $\zeta = \pm\infty$. Because $\phi_\infty = 0$ for bright solitons, the Eq. (13) can be expressed as

$$\theta' = \frac{v}{\delta_D}\left(1 - \frac{\phi_\infty^2}{\phi^2}\right) . \tag{14}$$

By using this formula, the equation of $\phi(\zeta)$ becomes

$$\phi'' = -\frac{\partial V}{\partial \phi} , \tag{15}$$

where

$$V(\phi) = \frac{1}{\delta_D}\left[\mu\, \phi^2 - \phi^2\sqrt{1 - \delta_N\, g\phi^2} + \frac{v^2\phi_\infty^4}{\delta_D}\frac{1}{\phi^2}\right] . \tag{16}$$

Thus, the field $\phi(\zeta)$ can be thought as the coordinate ϕ of a fictitious particle at time ζ. In this picture $V(\phi)$ is the external potential acting on the fictitious particle. The constant of motion is then given by

$$K = \frac{1}{2}\phi'^2 + V(\phi) \tag{17}$$

and after separation of variables we get

$$d\zeta = \frac{d\phi}{\sqrt{2\left(K - V(\phi)\right)}} . \tag{18}$$

We stress that in our previous papers [13b] and [14] only anomalous dispersion ($\delta_D = 1$) and $v\,\phi_\infty^2 = 0$ were considered. Here we investigate for the first time both normal dispersion ($\delta_D = -1$) and $v\,\phi_\infty^2 \neq 0$. In particular, the Eq. (14) and the centrifugal term of Eq. (16) are new and essential to study the case $v\,\phi_\infty^2 \neq 0$, i.e gray solitons with normal or anomalous dispersion.

3.1. Bright solitons

The NPSE admits bright solitons for $\delta_D \cdot \delta_N = 1$. In this case we have $\phi_\infty = 0$ and $K = 0$. By using the Eq. (18) we find that the bright soliton $\phi(\zeta)$ satisfies the implicit formula

$$\zeta = \frac{1}{\sqrt{2}}\sqrt{\frac{\delta_D}{1-\mu}}ArcTanh\left[\sqrt{\frac{\sqrt{1-\delta_N\, g\phi^2}-\mu}{1-\mu}}\right]$$
$$-\frac{1}{\sqrt{2}}\sqrt{\frac{\delta_D}{1+\mu}}ArcTan\left[\sqrt{\frac{\sqrt{1-\delta_N\, g\phi^2}-\mu}{1+\mu}}\right] , \tag{19}$$

where $ArcTanh[x]$ and $ArcTan[x]$ are the hyperbolic arctangent and the arctangent, respectively. They satisfy the formula $ArcTan[ix] = i\; ArcTanh[x]$ which is useful when $\delta_D = -1$. It is not difficult to show that this equation reduces to Eq. (7) in the 1D weak-coupling limit $g\phi^2 \ll 1$ taking into account the relationship between μ and g. This relationship is obtained by imposing the normalization condition to the bright soliton. Choosing $\int \phi(\zeta)^2 d\zeta = 1$ we obtain

$$g = \frac{2\sqrt{2}}{3}(2\mu+1)\sqrt{\frac{1-\mu}{\delta_D}} . \tag{20}$$

It is clear that for $\delta_D = -1$ it must be $\mu > 1$, while for $\delta_D = 1$ one has $0 < \mu < 1$ and the 3D bright soliton is dynamically stable only for $1/2 < \mu < 1$ [13, 14].

In Fig. 1 we plot the bright soliton with nonlinearity $g = 1$. The figure shows that the 1D bright soliton given by Eq. (7) and the two 3D bright solitons given by Eq. (18) have different axial profiles because their transverse profiles are quite different. For the 1D bright soliton $\sigma(\zeta) = 1$, while for the two 3D bright solitons $\sigma(\zeta)$ is not constant and depends on the sign δ_N of the nonlinearity: $\sigma(\zeta) \geq 1$ with $\delta_N = -1$ (self-defocusing and normal dispersion) and $\sigma(\zeta) \leq 1$ with $\delta_N = 1$ (self-focusing and anomalous dispersion).

In Fig. 2 we plot the axial width z_F and transverse width $\sigma(0)$ of the bright soliton as a function of the strength g. As expected, the axial width decreases by increasing g but for the A-F bright soliton, i.e. the 3D bright soliton with anomalous dispersion ($\delta_D = 1$) and self-focusing ($\delta_N = 1$), there is a critical strength $g_c = 4/3 = 1.3\bar{3}$ above which the soliton does not exist anymore (see also [5]). This critical value is in excellent agreeent with the numerical result $g_c = 1.35$ obtained by solving the 3D CNLSE [21].

For the N-D bright soliton, i.e. the 3D bright soliton with normal dispersion ($\delta_D = -1$) and self-defocusing ($\delta_N = -1$), there is not a critical strength and the N-D bright soliton exist for any g, as the strictly 1D bright soliton.

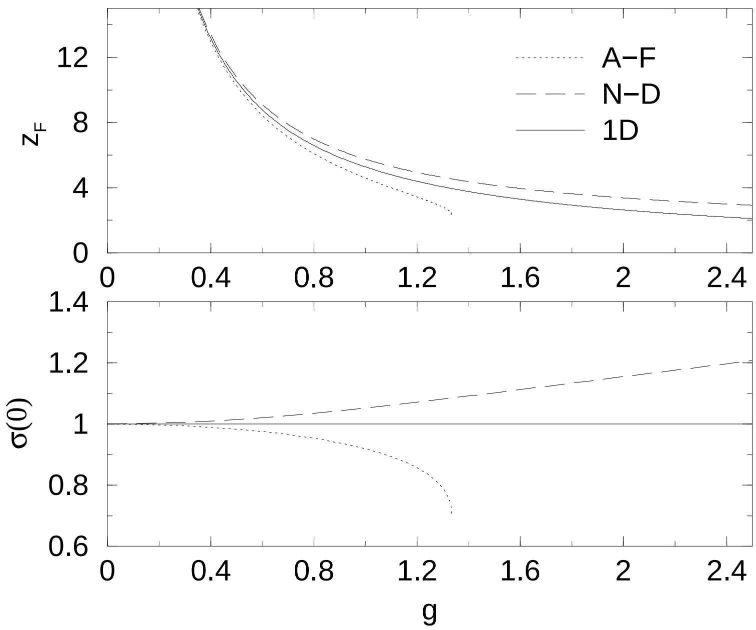

FIG. 2. Axial full width half maximum z_F and transverse width σ at $\zeta = 0$ as a function of the strength g of the nonlinearity for the bright soliton ($\delta_D \cdot \delta_N = 1$). The meaning of labels is the same of Fig. 1.

The lower panel of Fig. 2 shows that the transverse width of the N-D bright soliton grows indefinitely and a remarkable consequence is that the soliton geometry changes from cigar-shaped to disk-shaped by increasing the strength g of the cubic nonlinearity: at $g = 11.94$ we find $z_F = \sigma$.

3.2. Black and gray solitons

The NPSE admits dark solitons for $\delta_D \cdot \delta_N = -1$. In this case we have $\phi_\infty \neq 0$ and $K = V(\phi_\infty)$. By using the Eq. (17) we first numerically determine the dark soliton $\phi(\zeta)$ with $v = 0$. As previously stressed this dark soliton is called black soliton because its minimum value is zero. For the black soliton from Eq. (11) one finds $\theta' = 0$.

In Fig. 3 we plot the axial profile $\rho(\zeta) = \phi(\zeta)^2$ and the transverse width $\sigma(\zeta)$ of the black soliton setting $g = 1$ and $\phi_\infty = 1$. We compare the 1D black soliton given by Eq. (8) and the 3D black solitons obtained by numerically solving Eq. (15) with Eq. (16). The transverse width is $\sigma \simeq 1$ near the black hole but for 3D dark solitons it is $\sigma = (1 - \delta_N \phi_\infty^2)^{1/4}$ at infinity. It is clear that this condition bounds the domain of existence of the 3D dark soliton with self-focusing nonlinearity ($\delta_N = 1$): this dark soliton exist only for $g\phi_\infty^2 < 1$.

When $v \neq 0$ then the dark soliton is called gray soliton and its minimum value is greater than zero. In Fig. 4 we plot the hole of the 3D gray soliton with anomalous dispersion ($\delta_D = 1$) and self-defocusing nonlinearity ($\delta_N = 1$) for some values of v. By increasing the velocity v the depth of the hole is reduced and becomes zero when the

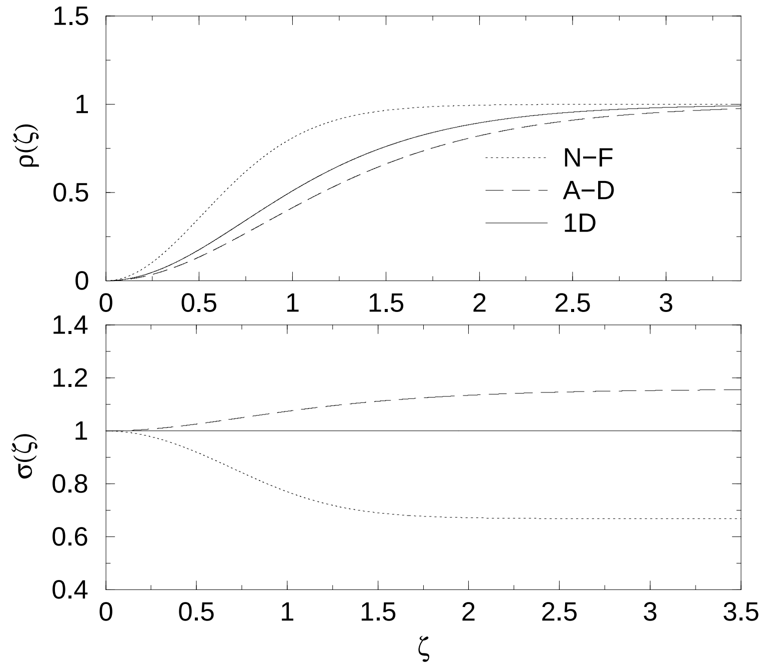

FIG. 3. Axial density profile $\rho(\zeta) = \phi(\zeta)^2$ and transverse width $\sigma(\zeta)$ of the dark soliton ($\delta_D \cdot \delta_N = -1$) with $v = 0$ (black soliton). Strength of the nonlinearty $g = 1$ and asymptotic field $\phi_\infty = 1$. A-D means 3D black soliton with anomalous dispersion ($\delta_D = 1$) and self-defocusing nonlinearity ($\delta_N = -1$); N-F means 3D black soliton with normal dispersion ($\delta_D = -1$) and self-focusing nonlinearity ($\delta_N = 1$).

velocity reaches the sound velocty c_s, given in general by the formula

$$c_s = \sqrt{ -\frac{5}{4} \frac{\delta_N g \rho_\infty}{\sqrt{1-\delta_N g \rho_\infty}} + \frac{1}{4} \frac{\delta_N g \rho_\infty - 2\delta_N^2 g^2 \rho_\infty^2}{(1-\delta_N g \rho_\infty)^{3/2}} } \, , \tag{21}$$

where $\rho_\infty = \phi_\infty^2$ [22]. The 3D gray soliton with $\delta_N = -1$ exist for any g if v is sufficiently small. Nevertheless, it has been shown [23] that these 3D dark solitons become dynamically unstable for a large nonlinearity g ($g = 1.5$ for the black soliton) due to the so-called snake instability. The snake instability implies the apparence of a non-axisymmetric purely imaginary eigenvalue in the Bogoliubov spectrum of elementary excitations. In this case the black soliton decays into a solitonic vortex [24].

In addition, it has been shown that many complex modes appear in the 3D black soliton at $g = 4$; for this nonlinearity an axisymmetric vortex ring emerges with energy intermediate between that of the black soliton and the solitonic vortex [24].

4. Conclusions

We have studied 3D bright and dark solitons with anomalous and normal dispersion taking into account their transverse width by using a non-polynomial Schrödinger equation. We have found new general analytical solutions for 3D bright solitons under transverse harmonic confinement. In addition, we have also analyzed 3D black and

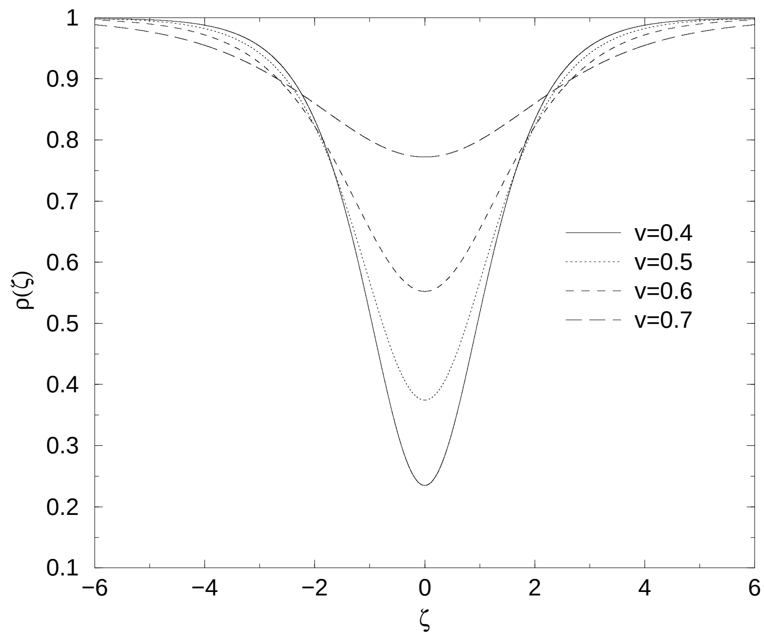

FIG. 4. Axial density profile $\rho(\zeta) = \phi(\zeta)^2$ of the gray soliton ($v \neq 0$) with anomalous dispersion ($\delta_D = 1$) and self-defocusing nonlinearity ($\delta_N = -1$). Strength of the nonlinearty $g = 1$ and asymptotic field $\phi_\infty = 1$.

gray solitons showing that self-focusing dark solitons (and with normal dispersion) exist only below a critical axial density. As a by-product we have obtained a formula for the sound velocity, i.e. the maximal velocity of propagation of a gray soliton, useful for both self-focusing and self-defocusing media. Atomic matter waves of Bose-Einstein condensates are now rutinely produced with dilute gases of alkali-metal atoms at ultralow temperatures. Also optical pulses in graded-index fibers are now available. As previously stressed, our predictions on 3D solitons under transverse confinement can be experimentally tested using these materials.

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